JOURNAL OF ALGEBRA **204**, 426–439 (1998)
ARTICLE NO. JA977265

Degree of Growth of Some Inverse Semigroups

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Communicated by T. E. Hall

Received January 28, 1997

We show that if a finitely presented Rees quotient of a free inverse semigroup has polynomial growth, then its growth series is in a certain subsemiring of $\mathbb{N}[[z]]$. We present a procedure for calculating the degree of growth of such a semigroup and use it to determine the possible values for the degree of growth. © 1998 Academic Press

1. INTRODUCTION

In [4] Shneerson and Easdown initiated the study of growth of finitely presented Rees quotients of free inverse semigroups. Growth was shown to be polynomial or exponential for semigroups from this class and an algorithm given to recognise which type of growth occurs [4, Section 3]. In [3], the author showed that if such a semigroup has polynomial growth, then it has rational growth series. The techniques in [3] are used to give more information about the form the growth series takes, and the possible values of the degree of growth.

Sections 2 and 4 recall definitions and results which relate the growth of a semigroup in our class with its word tree graph. Section 3 gives the definition and basic properties of the "degree of growth" of a series. In Section 5 we retrace the construction of the growth series of a semigroup from simpler series as presented in [3]. It is shown that the degree of growth can only take values in $\{0, 3, 4, 5, \dots\}$ and for each possible value, we give an example of a semigroup with that degree of growth.

0021-8693/98 $25.00

2. PRELIMINARIES

We assume familiarity with the basic definitions and elementary results from the theory of semigroups, which can be found in [1]. Let S be a semigroup generated by a finite subset X. Recall that the length $l(t)$ of an element $t \in S$ (*with respect to* X) is the least number of factors in all representations of t as a product of elements of X. Let

$$g_S(m) = |\{t \in S | l(t) \leq m\}|.$$

We say that S has *polynomial growth* if there exist natural numbers q and d such that

$$g_S(m) \leq qm^d,$$

for all natural numbers m, and *exponential growth* if there exists a real number $\alpha > 1$ such that

$$g_S(m) \geq \alpha^m,$$

for all sufficiently large m. It is clear that if S has polynomial [exponential] growth with respect to a given finite set of generators then it has polynomial [exponential] growth with respect to any finite set of generators.

The *growth series* of S with respect to X is defined to be the usual generating function for the sequence $g_S(m)$,

$$g_S(z) = \sum_{m=0}^{\infty} g_S(m) z^m.$$

However, as in [3] we shall work with a related series, namely, the generating function $h_S(z)$ of the sequence,

$$h_S(m) = |\{t \in S | l(t) = m\}|.$$

Because clearly $h_S(m) = g_S(m) - g_S(m-1)$, we have

$$h_S(z) = (1 - z) g_S(z).$$

Let A be a finite alphabet. We denote the free semigroup and the free inverse semigroup over A by F_A and FI_A, respectively. Equality in F_A is denoted by $\equiv$. Recall that a word w is *reduced* if w does not contain xx^{-1} as a subword for any letter $x \in A \cup A^{-1}$, and that w is *cyclically reduced* if w^2 is reduced (whence all powers of w are reduced).

Recall that elements of FI_A may be regarded as birooted word trees, the terminology and theory of which are explained in [1, Chapter 2] (see also [4, Section 2]). Thus if $u \in F_{A \cup A^{-1}}$, then regarded as an element of FI_A, u

may be represented as a birooted tree,

$$\varphi(u) = (T(u), \alpha(u), \beta(u)).$$

Recall that u is an idempotent of FI_A if and only if $\alpha(u) = \beta(u)$. Given a birooted word tree $B = (T, \alpha, \beta)$, we shall adopt the following notation:

(i) $e(B) = \max\{d(\alpha, v) | v \in T\}$,

(ii) for $m \in \mathbb{N}$, $B|_m = (T|_m, \alpha, \alpha)$, where $T|_m$ is the subtree of T obtained by deleting all vertices of distance greater than m away from α, and by deleting all edges incident on such vertices.

It is often convenient to adjoin an identity called 1 to FI_A and to adopt the convention that $l(1) = 0$ and $\varphi(1) = (\circ, \circ, \circ)$, the *null* birooted tree consisting of one vertex.

Given two trees $B_1 = (T_1, \alpha_1, \beta_1)$ and $B_2 = (T_2, \alpha_2, \beta_2)$, define $B_1 \oplus B_2 = (T, \alpha_1, \beta_2)$ where T is the word tree obtained by "pasting" T_1 and T_2 together, identifying β_1 with α_2 and then further identifying any isomorphic paths from the common vertex $\beta_1 = \alpha_2$. Note that $B_1 \oplus B_2 = \varphi(\varphi^{-1}(B_1)\varphi^{-1}(B_2))$.

A birooted word tree (T, α, β) on A is said to be *planted* if it is null or if it is nonnull and α is a leaf of the tree (that is, α is adjacent to exactly one vertex of T). We shall refer to a birooted word tree which is planted simply as a *planted tree*. Given two planted trees $P_1 = (T_1, \alpha_1, \beta_1)$ and $P_2 = (T_2, \alpha_2, \beta_2)$, P_1 is a *subplanted tree* of P_2 if there is a word tree monomorphism $T_1 \to T_2$ sending α_1 to α_2.

Given as nonnull planted tree $P = (T, \alpha, \beta)$, we shall adopt the following notation:

(i) $\gamma(P)$ is the unique vertex of T adjacent to α,

(ii) the label of P, denoted by label(P), is the label of the edge $\alpha \to \gamma(P)$.

Let $P_1 = (T_1, \alpha_1, \alpha_1)$ and $P_2 = (T_2, \alpha_2, \alpha_2)$ be two planted trees representing idempotents, with $P_1 \neq \varphi(1)$. If $P_2 = \varphi(1)$, or label$(P_1) \neq$ label$(P_2)^{-1}$, then define $P_1 \odot P_2$ to be the planted word tree (T, α_1, α_1), where T is the word tree obtained by pasting T_1 and T_2 together, identifying $\gamma(P_1)$ with α_2 and then further identifying any isomorphic paths from the common vertex $\gamma(P_1) = \alpha_2$.

Let P be a planted tree representing an idempotent with label x. Then P can be uniquely expressed in the form,

$$P = \varphi(x) \oplus P_1 \oplus \cdots \oplus P_s \oplus \varphi(x^{-1}),$$

where $s \geq 0$, $P_1, \ldots, P_s$ are nonnull planted trees with distinct labels, and for each i, P_i represents an idempotent and $\text{label}(P_i) \neq x^{-1}$. We shall call $P_1, \ldots, P_s$ and the null tree the *components* of P.

3. DEGREE OF GROWTH OF A SERIES

Let $a(z) = \sum_{i \geq 1} a_i z^i \in \mathbb{N}[[z]]$. If $a_m \geq 1$ for infinitely many m, then define

$$L(a(z)) = \{\alpha \in \mathbb{R} | \text{there exists } q > 0 \text{ such that } a_m \leq q m^\alpha \text{ for all } m\},$$

and define the *degree of growth* of $a(z)$ to be

$$\deg(a(z)) = \inf L(a(z)).$$

So $\deg(a(z)) \geq 0$ and if $L(a(z))$ is empty, $\deg(a(z)) = \infty$. If $\deg(a(z))$ is defined, it is a routine exercise to show that

$$\deg(a(z)) = \limsup \log_m a_m.$$

It is convenient to adopt the convention that $\deg(a(z)) = -1$ if $a(z) \in \mathbb{N}[z]$. The following properties are easy to verify:

(3.1) LEMMA. (i) *if* $a(z), b(z) \in \mathbb{N}[[z]]$ *are such that* $\deg(a(z))$, $\deg(b(z))$ *are defined, then* $\deg(a(z) + b(z))$ *is defined and equals* $\max\{\deg(a(z)), \deg(b(z))\}$.

(ii) *If* $\deg(a(z))$ *is defined, and* $0 \neq b(z) \in \mathbb{N}[z]$, *then* $\deg(a(z)b(z))$ *is defined and equals* $\deg(a(z))$.

Let $\sum a_m z^m = 1/(1-z)^k$. Then

$$a_m = \binom{m+k-1}{k-1} = \frac{(m+(k-1))(m+(k-2))\cdots(m+1)}{(k-1)!}.$$

Therefore $\deg(1/(1-z)^k) = k-1$, and the infimum in the definition of deg is attained. Using part (ii) of the previous lemma we see that for any positive integers $s_1, \ldots, s_k$,

$$\deg\left(\prod_{i=1}^{k} \frac{1}{1-z^{s_i}}\right) = \deg\left(\prod_{i=1}^{k} (1 + z + \cdots + z^{s_i - 1}) \prod_{i=1}^{k} \frac{1}{1-z^{s_i}}\right)$$
$$= \deg\left(\frac{1}{(1-z)^k}\right) = k-1.$$

Denote by $\mathfrak{R}$ the semiring consisting of series $f(z) \in \mathbb{N}[[z]]$ which have the form,

$$f(z) = \sum_{i=1}^{l} \frac{p_i(z)}{q_i(z)},$$

where for each i, $p_i(z) \in \mathbb{N}[z] \setminus \{0\}$ and $q_i(z) = \prod_{j=1}^{k_i}(1 - z^{s_{ij}})$ for some integers $k_i \geq 0$, $s_{ij} > 0$. With the preceding observations it is easy to prove the following

(3.2) PROPOSITION. *If* $a(z) \in \mathfrak{R}$, *then* $\deg(a(z))$ *is defined and is an integer. Moreover,* $d = \deg(f(z)) \in L(f(z))$, *that is, there exists* $q > 0$ *such that* $a_m \leq qm^d$ *for all* m. *If* $f_1(z), \ldots, f_s(z) \in \mathfrak{R}$, *then*

$$\deg\big(f_1(z) \cdots f_s(z)\big) = -1 + \sum_{i=1}^{s} \big(\deg(f_i(z)) + 1\big).$$

Lemma (3.1) and Proposition (3.2) will be used frequently in the rest of the paper without comment.

We are of course mainly interested in $\deg(g_S(z))$ for some finitely generated semigroup S. It should be noted that the definition of $\deg(g_S(z))$ is analogous to the *Gelfand–Kirillov dimension* of a finitely generated unital algebra over a field, as defined in [2, Chapter 1]. The following is a routine exercise, and has its analogue for algebras in [2, Chapter 1].

(3.3) PROPOSITION. *Let* S *be a finitely generated semigroup. Then*

(i) $\deg(g_S(z))$ *is independent of the choice of generators.*

(ii) *If* T *is a finitely generated subsemigroup of* S, *then* $\deg(g_T(z)) \leq \deg(g_S(z))$.

4. GRAPHS AND POLYNOMIAL GROWTH

In this section we review graphical conditions for a Rees quotient of a finitely generated free inverse semigroup by a finitely generated ideal to have polynomial growth.

Let $\mathfrak{M}_{FI}$ denote the class of finitely presented inverse semigroups S with zero having a presentation of the form,

$$S = \langle A | c_i = 0 \text{ for } i = 1, \ldots, k \rangle,$$

where A is some finite alphabet, k is some nonnegative integer, $c_i \in F_{A \cup A^{-1}}$ for $i = 1$ to k. Then $\mathfrak{M}_{FI}$ is precisely the class of Rees quotients of finitely generated free inverse semigroups by finitely generated ideals. We shall write $A \cup A^{-1} = \{x_1, x_2, \ldots, x_{2n}\}$ and we shall put

$$d + 1 = \max\{3, |c_i| \,|i = 1, \ldots, k\},$$

where $|c_i|$ denotes the length of c_i with respect to $F_{A \cup A^{-1}}$. By our definition, $d \geq 2$. We have the following useful criterion [4, Lemma 3.1].

(4.1) LEMMA. *Let $w \in F_{A \cup A^{-1}}$. Then $w = 0$ in S if and only if $T(w)$ contains $T(c_i)$ as a subtree for some i.*

Shneerson and Easdown introduced in [4] a directed graph Γ_S, the *Ufnarovsky graph of S* (depending on the presentation of S). Vertices of Γ_S are defined to be reduced words of length d which are nonzero in S. If v_1 and v_2 are vertices then a directed edge from v_1 to v_2 is defined in Γ_S if there exist letters $g, h \in A \cup A^{-1}$ such that $v_1 g$ is a reduced word which is nonzero in S and $v_1 g \,\overline{\overline{\circ}}\, h v_2$. We regard the letter g as a label for this edge. Paths in Γ_S may then be labelled by reduced words which are nonzero in S. Growth was shown to be polynomial or to be exponential for semigroups in $\mathfrak{M}_{FI}$ and a criterion on Γ_S is given to recognise which type of growth occurs [4, Section 3].

Another directed graph Γ'_S, called the *word tree graph of S* is introduced in [3].

(i) the vertices of Γ'_S are planted trees $P = (T, \alpha, \alpha)$ on A satisfying the properties that $e(P) = d$ and $\varphi^{-1}(P)$ is nonzero in S;

(ii) there is a directed edge from vertex P to vertex Q if $Q|_{d-1}$ is a component of P and $\varphi^{-1}(P \odot Q)$ is nonzero in S.

Note that if $Q|_{d-1}$ is a component of P, then $\mathrm{label}(P) \neq \mathrm{label}(Q)^{-1}$, so $P \odot Q$ is defined. Furthermore $(P \odot Q)|_d = P$.

Denote the set of vertices of Γ'_S by $V(\Gamma'_S)$. For $P \in V(\Gamma'_S)$, define

$$V(P) = \{P' \in V(\Gamma'_S) | P \to P' \text{ is an edge in } \Gamma'_S\},$$

$$V_i(P) = \{P' \in V(P) | \mathrm{label}(P') = x_i\}, \quad \text{for } i = 1, \ldots, 2n.$$

Here the label of a vertex in Γ'_S is simply its label as a planted tree.

Using the word tree graph we can give another characterisation of polynomial growth [3, Theorem 5.3].

(4.2) THEOREM. *Let $S = \langle A | c_i = 0 \text{ for } i = 1, \ldots, k\rangle$ be an inverse semigroup from the class $\mathfrak{M}_{FI}$. Then S has polynomial growth if and only if Γ'_S has no vertex contained in different cycles.*

In fact, polynomial growth restricts the structure of Γ'_S more than the previous theorem suggests:

(4.3) THEOREM. *Suppose S has polynomial growth. If $P \in V(\Gamma'_S)$ is in a cycle, then $P = \varphi(ww^{-1})$ for some reduced word w.*

Proof. Let $P = P_1 \to \cdots \to P_s \to P$ be the cycle. Let $u = a_1 a_2 \cdots a_s$ where $a_i = \text{label}(P_i)$. Then it is clear that u is a cyclically reduced word and if w is the prefix of length d of u^d, then $\varphi(ww^{-1})$ is a subplanted tree of P.

Suppose $P \neq \varphi(ww^{-1})$. Then there are nonempty words v_1, v_2 such that $w \,\overline{\underline{\circ}}\, v_1 v_2$ and a letter $z \in A \cup A^{-1}$, such that $v_1 z$ and $z^{-1} v_2$ are reduced words and

$$\varphi\Big(\big(v_1 z z^{-1} v_2\big)\big(v_1 z z^{-1} v_2\big)^{-1}\Big)$$

is a subplanted tree of P.

In fact, it is easy to see that there is a cyclically reduced word v such that $z^{-1} v z$ is reduced and for each $l \in \mathbb{N}$, $T((z^{-1} v z)^l)$ is a subtree of

$$\big(P_1 \odot \big(P_2 \odot \cdots \odot \big(P_s \odot \big(P_1 \odot \big(P_2 \odot \cdots \odot \big(P_s \odot \cdots\big) \cdots\big)\big)\big) \cdots\big)\big),$$

provided the $\odot$ operation is iterated enough times. In particular $(z^{-1} v z)^l$ is nonzero in S. This implies S has exponential growth by [4, Lemma 3.2], which is a contradiction. ∎

Before studying $\deg(g_S(z))$ in detail in the next section, the following proposition shows that either $\deg(g_S(z)) = 0$ or $\deg(g_S(z)) \geq 3$.

(4.4) PROPOSITION. *Suppose S has polynomial growth. Then the following conditions are equivalent*:

(a) *S is infinite.*

(b) *Γ_S contains a cycle.*

(c) *Γ'_S contains a cycle.*

(d) *S contains a free monogenic inverse subsemigroup.*

(e) $\deg(g_S(z)) \geq 3$.

Proof. Suppose Γ_S does not contain a cycle. Let N be the length of the longest path in Γ_S. If $a_1 \cdots a_l$ is a reduced word of length $l \geq d$ and is nonzero in S, then

$$a_1 \cdots a_d \to a_2 \cdots a_{d+1} \to \cdots \to a_{l-d+1} \cdots a_l$$

is a path in Γ_S. Therefore $l - d + 1 \leq N$, and $l \leq N + d - 1$. It follows that if a word u is nonzero in S then $e(\varphi(u)) \leq N + d - 1$. This shows that S is finite. Therefore (a) implies (b).

Suppose (b) holds and $w_1 \to w_2 \to \cdots \to w_s \to w_1$ is a cycle in Γ_S. Then

$$\varphi\left(w_1 w_1^{-1}\right) \to \varphi\left(w_2 w_2^{-1}\right) \to \cdots \to \varphi\left(w_s w_s^{-1}\right) \to \varphi\left(w_1 w_1^{-1}\right)$$

in a cycle in Γ'_S. Therefore (b) implies (c). The converse is clear by using Theorem (4.3). Therefore (b) and (c) are equivalent.

If w is the word which labels a cycle in Γ_S, then w is cyclically reduced and w^k is nonzero in S for all positive integers k. Hence w generates a free monogenic inverse subsemigroup of S. Therefore (b) implies (d).

It is shown in [3, Section 3] that if $T = FI^1_{\{a\}}$, then

$$h_T(z) = \frac{1}{(1-z^2)(1-z)^2}.$$

Because $g_T(z) = 1/(1-z)h_T(z)$, $\deg(g_T(z)) = 3$. Therefore Proposition (3.3) shows that (d) implies (e).

If S is finite, then $h_S(z) \in \mathbb{N}[z]$ and $\deg(g_S(z)) = \deg(h_s(z)/(1-z))$ $= 0$. Therefore (e) implies (a). ∎

The equivalence of (a) and (b) was observed in a remark at the end of [4, Section 3].

5. DEGREE OF GROWTH OF S

Throughout this section let S denote an inverse semigroup from the class $\mathfrak{M}_{FI}$ which has polynomial growth. We shall retrace our construction of $h_S(z)$ from simpler series, which is described in [3, Section 4]. As we shall see, Theorem (4.3) is crucial in simplifying the calculations that arise. First we review some definitions.

For each vertex P in Γ'_S define $A(P)$ to be the set of $u \in FI_A$ such that u is nonzero in S, and $\varphi(u)$ is a planted word tree with $\varphi(u)|_d = P$. For $r, m \in \mathbb{N}$ put

$$A(P, r, m) = \{u \in A(P) | l(u) = m, d(\alpha(u), \beta(u)) = r\},$$

and

$$B(P, m) = \{u \in A(P) | l(u) = m, d(\alpha(u), \beta(u)) > 0\}.$$

Let $a(P, r, m) = |A(P, r, m)|$ and $b(P, m) = |B(P, m)|$. Denote their generating functions by $a(P, r)(z)$ and $b(P)(z)$, respectively. For convenience we shall write $a(P)(z)$ for $a(P, 0)(z)$.

Let P be a vertex of Γ'_S with label x_j. Then P can be uniquely expressed in the form

$$P = \varphi(x_j) \oplus P_1 \oplus \cdots \oplus P_{2n} \oplus \varphi\left(x_j^{-1}\right),$$

where P_k is null for k such that $x_k = x_j^{-1}$ and, for each i, either P_i is null or P_i is a planted tree with label x_i. For $1 \le i \le 2n$, let $c(P, i, r, m)$ denote the number of planted trees $B = (T, \alpha, \beta)$ such that $(T, \alpha, \alpha) = P_i$, $d(\alpha, \beta) = r$, and $l(\varphi^{-1}(B)) = m$. Define the polynomials,

$$c(P, i, r)(z) = \sum_{m \ge 0} c(P, i, r, m) z^m.$$

Because $e(P_i) < d$ for each i, it is clear that $c(P, i, r)(z) = 0$ whenever $r \ge d$.

(5.1) LEMMA. *For each vertex P in Γ'_S, $a(P)(z) \in \mathfrak{R}$. If P is not in a cycle, then*

$$\deg(a(P)(z)) = -1 + \sum_{i=1}^{2n} \max\{0, \deg(a(P')(z)) + 1 | P' \in V_i(P)\}.$$

If P is in a cycle $P = P_1 \to \cdots \to P_s \to P$, then

$$\deg(a(P)(z)) \\ = \max\left\{0, \deg(a(P')(z)) + 1 | P' \in \bigcup_{i=1}^{s} V(P_i) \setminus \{P_1, \ldots, P_s\}\right\}.$$

Proof. Recall that the *strongly connected components* of a directed graph are the equivalence classes of vertices under the "are mutually reachable" relation. By Theorem (4.2), a strongly connected component of Γ'_S consists of either one vertex or one cycle.

Let $C_1, \ldots, C_t$ be the strongly connected components of Γ'_S, ordered so that the vertices in C_i are reachable from the vertices in C_j only if $i \le j$. We shall prove the lemma by induction. So suppose the lemma holds for all vertices in $C_1 \cup \cdots \cup C_{k-1}$. (So $k = 1$ initially.)

By Proposition (4.2) of [3], for each vertex P in Γ'_S,

$$a(P)(z) = z^2 \prod_{i=1}^{2n} \left\{c(P, i, 0)(z) + \sum_{P' \in V_i(P)} a(P')(z)\right\}.$$

Suppose $C_k = \{P\}$. Then $a(P')(z) \in \mathfrak{R}$ for each $P' \in V(P)$ by the inductive hypothesis. Therefore $a(P)(z) \in \mathfrak{R}$. The formula follows easily from Lemma (3.1) and Proposition (3.2).

If C_k is a cycle $P = P_1 \to \cdots \to P_s \to P_{s+1} = P$, then for each i, $P_i = \varphi(w_i w_i^{-1})$ for some reduced word w_i by Theorem (4.3). Let the second letter of w_i be x_{j_i}. Then $c(P_i, j, 0)(z) = 1$ and $V_j(P_i) = \varnothing$ if $j \neq j_i$. So

$$a(P_i)(z) = z^2\left\{c(P_i, j_i, 0)(z) + \sum_{P' \in V(P_i)} a(P')(z)\right\}$$
$$= z^2 a(P_{i+1})(z) + z^2 \sum_{P' \in V(P_i)\setminus C_k} a(P')(z) + z^2 c(P_i, j_i, 0)(z),$$

because $V(P_i) \setminus \{P_{i+1}\} = V(P_i) \setminus C_k$ by Theorem (4.2). By substitution we get

$$(1 - z^{2s})a(P)(z) = p(z) + \sum_{i=1}^{s} \sum_{P' \in V(P_i)\setminus C_k} z^{2i} a(P')(z),$$

for some $p(z) \in \mathbb{N}[z]$. So again the result follows. ∎

By Proposition (4.2) of [3], $za(P,1)(z) = a(P)(z)$ for each vertex P in Γ'_S. Therefore $a(P,1)(z) \in \mathfrak{R}$ by Lemma (5.1), and $\deg(a(P,1)(z)) = \deg(a(P)(z))$ by Lemma (3.1). Also by Proposition (4.2) of [3],

$$a(P, r)(z) = z \sum_{j=1}^{2n} \left\{c(P, j, r-1)(z) + \sum_{P' \in V_j(P)} a(P', r-1)(z)\right\}$$
$$\times \prod_{i \neq j} \left\{c(P, i, 0)(z) + \sum_{P' \in V_i(P)} a(P')(z)\right\},$$

for each $r \geq 2$. Therefore,

$$b(P)(z) = \sum_{r \geq 1} a(P, r)(z)$$
$$= a(P, 1)(z) + z \sum_{j=1}^{2n} \left\{c(P, j)(z) + \sum_{P' \in V_j(P)} b(P')(z)\right\}$$
$$\times \prod_{i \neq j} \left\{c(P, i, 0)(z) + \sum_{P' \in V_i(P)} a(P')(z)\right\},$$

where $c(P, j)(z) = \sum_{r \geq 1} c(P, j, r)(z) \in \mathbb{N}[z]$.

Given this formula for $b(P)(z)$, we can again deduce a formula for $\deg(b(P)(z))$, which fortunately simplifies to the following

(5.2) LEMMA. *For each vertex P in Γ'_S, $b(P)(z) \in \mathfrak{R}$. If a cycle is reachable from P, then* $\deg(b(P)(z)) = \deg(a(P)(z)) + 1$. *Otherwise* $\deg(b(P)(z)) = \deg(a(P)(z)) = -1$.

Proof. We shall use induction as in the proof of Lemma (5.1). Again suppose the lemma holds for all vertices in $C_1 \cup \cdots \cup C_{k-1}$. (So $k = 1$ initially.)

Suppose $C_k = \{P\}$. If no cycle is reachable from P, then $b(P)(z)$ and $a(P)(z)$ are polynomials. So the result is clear.

Now suppose a cycle is reachable from P. Because P is not in a cycle, a cycle is reachable from P_0 for some $P_0 \in V(P)$. Now by induction, $\deg(b(P_0)(z)) = \deg(a(P_0)(z)) + 1$, $\deg(b(P')(z)) \le \deg(a(P')(z)) + 1$ and $b(P')(z) \in \mathfrak{R}$ for all $P' \in V(P)$. Therefore, by Lemma (5.1) and the observations preceding Lemma (5.2), $b(P)(z) \in \mathfrak{R}$, and

$$\deg(b(P)(z)) = \deg(a(P)(z)) + 1,$$

by comparing the formulae for $a(P)(z)$ and $b(P)(z)$.

Suppose that C_k is a cycle $P = P_1 \to \cdots \to P_s \to P_{s+1} = P$. Then, as in the proof of the previous lemma, for each i there is some j_i, such that $c(P_i, j, 0)(z) = 1$, $c(P_i, j)(z) = 0$, and $V_j(P_i) = \varnothing$ if $i \ne j_i$. So

$$b(P_i)(z) = a(P_i, 1)(z) + z\left\{c(P_i, j_i)(z) + \sum_{P' \in V(P_i)} b(P')(z)\right\}.$$

By substitution we get

$$(1 - z^s)b(P)(z)$$
$$= p(z) + \sum_{i=1}^{s} z^{i-1} a(P_i, 1)(z) + \sum_{i=1}^{s} \sum_{P' \in V(P_i)\setminus C_k} z^i b(P')(z),$$

for some $p(z) \in \mathbb{N}[z]$. Therefore $b(P)(z) \in \mathfrak{R}$ by induction. Now for all $P' \in V(P_1) \cup \cdots \cup V(P_s) \setminus \{P_1, \ldots, P_s\}$,

$$\deg(b(P')(z)) \le \deg(a(P')(z)) + 1,$$

by induction, and

$$\deg(a(P')(z)) + 1 \le \deg(a(P)(z)),$$

by Lemma (5.1), so

$$\deg(b(P')(z)) \le \deg(a(P)(z)).$$

Also by Lemma (5.1), $\deg(a(P)(z)) = \deg(a(P_i)(z)) = \deg(a(P_i, 1)(z))$ for each i. So

$$\deg((1 - z^s)b(P)(z)) = \deg(a(P)(z)),$$

and the formula follows by Proposition (3.2). ∎

These lemmas enable us to calculate $\deg(h_S(z))$. Recall from the proof of Proposition (4.4) of [3] that $h_S(z)$ can be expressed as a sum of terms of the form,

$$p(z)a(P_1)(z)a(P_2)(z)\cdots a(P_s)(z),$$

or, for some $1 \le j \le s$,

$$p(z)b(P_j)(z)\prod_{i\neq j} a(P_i)(z),$$

where $P_1, \dots, P_s$ are vertices in Γ'_S with distinct labels such that $\varphi^{-1}(P_1 \oplus \cdots \oplus P_s)$ is nonzero in S, and $p(z)$ is some polynomial in $\mathbb{N}[z]$.

If $\deg(a(P_i)(z)) = \deg(b(P_i)(z)) = -1$, then the degree of growth of either one of the foregoing series is unchanged by leaving off the $a(P_i)(z)$ or $b(P_i)(z)$ term. Because, by Lemma (5.2), $\deg(b(P)(z)) = \deg(a(P)(z)) + 1$ for some vertex P in Γ'_S if S is infinite, we have

(5.3) PROPOSITION. *If S is infinite, then* $\deg(h_S(z))$ *is the maximum value of*

$$1 + \deg\big(a(P_1)(z)a(P_2)(z)\cdots a(P_s)(z)\big),$$

where $P_1, \dots, P_s \in V(\Gamma'_S)$ have distinct labels, a cycle is reachable from P_i for each i, and $\varphi^{-1}(P_1 \oplus \cdots \oplus P_s)$ is nonzero in S.

Combining Lemma (5.1) and Proposition (5.3) gives a very easy algorithm for calculating $\deg(h_S(z))$. We now give a family of examples in which every possible degree of growth occurs.

For each positive integer n, define the inverse semigroup,

$$S = \langle x_1, \dots, x_n | x_i x_j = 0 \text{ if } j \neq i, i-1,\ x_i^{-1}x_j = x_i x_j^{-1} = 0 \text{ if } j \neq i \rangle.$$

Note that when $n = 1$, S is the monogenic free inverse semigroup with zero.

By Lemma (4.1), it is easy to see that if a word tree contains a vertex which is an endpoint for more than two edges, then it represents a word which is zero in S. Therefore the word tree graph of S is the same as the Ufnarovsky graph of S, depicted in Fig. 1, by identifying the vertex w with $\varphi(ww^{-1})$.

Using Lemma (5.1), we have

$$\deg\big(a(x_i^2)(z)\big) = i - 1, \qquad \text{for } i = 1, 2, \dots, n.$$

$$\deg\big(a(x_i^{-2})(z)\big) = n - i, \qquad \text{for } i = 1, 2, \dots, n.$$

$$\deg\big(a(x_{i+1}x_i)(z)\big) = i - 1, \qquad \text{for } i = 1, 2, \dots, n-1.$$

$$\deg\big(a(x_i^{-1}x_{i+1}^{-1})(z)\big) = n - i - 1, \qquad \text{for } i = 1, 2, \dots, n-1.$$

$$\begin{tikzcd}
x_n^2 \arrow[loop above] \arrow[r] & x_n x_{n-1} \arrow[r] & x_{n-1}^2 \arrow[loop above] \arrow[r] & \cdots \arrow[r] & x_2 x_1 \arrow[r] & x_1^2 \arrow[loop above]
\end{tikzcd}$$

$$\begin{tikzcd}
x_1^{-2} \arrow[loop above] \arrow[r] & x_1^{-1} x_2^{-1} \arrow[r] & x_2^{-2} \arrow[loop above] \arrow[r] & \cdots \arrow[r] & x_{n-1}^{-1} x_n^{-1} \arrow[r] & x_n^{-2} \arrow[loop above]
\end{tikzcd}$$

FIGURE 1

Using Lemma (4.1), we can see that if $P_1, P_2 \in V(\Gamma_S')$ have distinct labels, then $\varphi^{-1}(P_1 \oplus P_2)$ is nonzero in S if and only if

$$\{\text{label}(P_1), \text{label}(P_2)\} = \{x_i, x_i^{-1}\} \text{ or } \{x_i, x_{i+1}^{-1}\},$$

for some i. Now

$$\begin{aligned}
\deg\big(a\big(x_{i+1}^{-2}\big)(z)\big) &< \deg\big(a\big(x_i^{-2}\big)(z)\big), && \text{for } i = 1, 2, \ldots, n-1. \\
\deg\big(a\big(x_i^{-1}x_{i+1}^{-1}\big)(z)\big) &< \deg\big(a\big(x_i^{-2}\big)(z)\big), && \text{for } i = 1, 2, \ldots, n-1. \\
\deg\big(a\big(x_{i+1}^{-1}x_{i+2}^{-1}\big)(z)\big) &< \deg\big(a\big(x_i^{-2}\big)(z)\big), && \text{for } i = 1, 2, \ldots, n-2. \\
\deg\big(a(x_i x_{i-1})(z)\big) &< \deg\big(a\big(x_i^{2}\big)(z)\big), && \text{for } i = 2, 3, \ldots, n.
\end{aligned}$$

Therefore by Proposition (5.3) and (3.2),

$$\begin{aligned}
\deg(h_S(z)) &= \max\big\{1 + \deg\big(a\big(x_i^{-2}\big)(z)a\big(x_i^{2}\big)(z)\big)\big| i = 1, 2, \ldots, n\big\} \\
&= \max\big\{2 + \deg\big(a\big(x_i^{-2}\big)(z)\big) + \deg\big(a\big(x_i^{2}\big)(z)\big)\big| i = 1, 2, \ldots, n\big\} \\
&= \max\{2 + (n-i) + (i-1) | i = 1, 2, \ldots, n\} \\
&= n + 1.
\end{aligned}$$

To summarize, we show

(5.4) THEOREM. *Suppose an inverse semigroup S from the class $\mathfrak{M}_{FI}$ has polynomial growth. Then $h_S(z) \in \mathfrak{R}$. The possible values for* $\deg(h_S(z))$ *are* $\{-1, 2, 3, 4, \ldots\}$.

Because $h_S(z) = (1 - z)g_S(z)$, $g_S(z) \in \mathfrak{R}$ also, and the possible values for $\deg(g_S(z))$ are $\{0, 3, 4, 5, \ldots\}$.

Of course, the fact that $g_S(z) \in \mathfrak{R}$ implies that $\deg(g_S(z)) \in L(g_S(z))$ (see Proposition (3.2)), and the radius of convergence of $g_S(z)$ is 1.

Remark. The author was told that L. M. Shneerson also proved, by a different method, that the possible values for $\deg(g_S(z))$ are $\{0, 3, 4, 5, \ldots\}$.

REFERENCES

1. P. M. Higgins, "Techniques of Semigroup Theory," Oxford Univ. Press, 1992.
2. G. R. Krause and T. H. Lenagan, "Growth of Algebras and Gelfand–Kirillov Dimension," Research Notes in Math., Vol. 116, Pitman, Boston, 1985.
3. J. Lau, "Rational Growth of a Class of Inverse Semigroups," Research Report of the School of Mathematics and Statistics, University of Sydney, (96)31.
4. L. M. Shneerson and D. Easdown, Growth and existence of identities in a class of finitely presented inverse semigroups with zero, *Internat. J. Algebra Comput.* **6**(1) (1996), 105–121.